**Applications of topology**

**Applications to Digital Image Processing:** Nowadays, digital images are becoming one of the key means of communication while dealing with graphical information. The images in a digital camera, the demonstrations of words in a book, artworks and graphics are the examples of digitalimages. The area of digital image processing deals with the formation, storage, operation and demonstration of digital images. For every phase of digital image processing, topological conceptsand tools are involved to solve some questions or problems. The main role is played by Digital topology, where the topological setting is the digital plane, which is the space acquired by taking the product of two digital lines, where the digital line is the set of integers 𝑍. The basis element for each odd integer is given by (𝑛) = {𝑛}, and the integers are termed as pixels. Therefore, every single pixel is an open set in digital topology. Digital topology as such is the study of topological interactions with digital image display. The digital plane in digital topology is the topological space 𝑍 × , and the subspace of digital plane which consists of all the open points is termed as visible screen.

𝐹𝑜𝑟 𝑒𝑣𝑒𝑟𝑦 (𝑚, 𝑛) ∈ 𝑍 × 𝑍, 𝑏𝑎𝑠𝑖𝑠 𝑒𝑙𝑒𝑚𝑒𝑛𝑡 𝑖𝑠 𝑔𝑖𝑣𝑒𝑛 𝑎𝑠: 𝐵(𝑚, 𝑛) = { {(𝑚, 𝑛)} 𝑖𝑓 𝑚 𝑎𝑛𝑑 𝑛 𝑎𝑟𝑒 𝑜𝑑𝑑 {(𝑚 + 𝑎, 𝑛)|𝑎 = −1,0,1} 𝑖𝑓 𝑚 𝑖𝑠 𝑒𝑣𝑒𝑛 𝑎𝑛𝑑 𝑛 𝑖𝑠 𝑜𝑑𝑑 {(𝑚, 𝑛 + 𝑏)|𝑏 = −1,1,0} 𝑖𝑓 𝑚 𝑖𝑠 𝑜𝑑𝑑 𝑎𝑛𝑑 𝑛 𝑖𝑠 𝑒𝑣𝑒𝑛 {(𝑚 + 𝑎, 𝑛 + 𝑏)|𝑎, 𝑏 = −1,1,0} 𝑖𝑓 𝑚, 𝑛 𝑎𝑟𝑒 𝑒𝑣𝑒n

**Application to Robotics:** Topology and physics have a very deep connection. It requires the most advanced knowledge of topology to study the most sophisticated applications in other fields. In physics, the first or basic construction is of Configuration space, which acts as the topological space over there. For studying configuration space, we have to have some track of variables that are related to position and arrangement of the objects involved. Suppose we may need to keep the track of various parts of the robot arm. So the the configuration space which acts as the topological space facilitates us to achieve this tracking of variables. Also for the investigation of other perceptions like momentum, velocity etc. of the system, we are left with one more space called phase space. Moreover, topology as being very influential for functions, there is good number of maps defined in topology, one natural map is forward kinematics map, which is very valuable in motion design for ties, robot arms and other related machineries. The forward kinematics map gives critical information about unmanageable, obstructive, or possibly problematic arrangements of a mechanism we are considering. Generally, while studying the robot arms or machine designs, we get to work with a particular point in the mechanism where a function comes into picture as a tool to pick up a fragment, drill a hole, spray paint etc. This particular point is known as end effector. Here every point of the configuration space of a mechanism is then sent to their respective end effector points of the operational space. Therefore, a function 𝑓 is assigned, which is referred as the forward kinematics map. The continuity of 𝑓 is obvious because the points nearby in the configuration space are mapped to the points nearby in the operational space.

**Applications to Biology:** Topology besides being a very different branch of mathematics, it plays a very good role in Biology also. Since genotypes-phenotypes are of very primary importance in biology, we see how topology is even useful in sequencing the right nucleotides in DNA strand. Genotypes are the internally veiled and inheritable information of a living being while as phenotypes are the physical appearances of that information. Topology solves one most important problem in DNA research. As we know that DNA is composed of four nucleotides: Adenine, Cytosine, Guanine and Thymine. These are arranged in a manner that they resemble a sequence. The sequence of nucleotides on every single chain of DNA decides the sequence of the other chain. The problem found in DNA research is in the comparison of distinct DNA sequences. In topology we define something called Metric, which is basically a distance function used to measure the distance between the elements of a set. Therefore the sets on which a metric can be defined are called metric spaces.

**Applications in Civil Engineering:** Topology has got applications to Civil engineering also like we have one in Bridge Design. Bridge Design is given by Topological Optimisation Technology by Zhi Hao Zuo et.al. (2018) [3]. Topology optimisation is better to aim for locations and shapes of cavities in the design area. There are several number of topology optimisation but the evolutionary structural optimisation (ESO) is mostly used due to its simplicity in software implementation. In this paper this technique is executed and presented in its structural design, where the particular focus is on the design of several bridges. In this design application, various constructional requirements are involved which includes support types and selection of the elevation. Though the requirements are not less, and also during design problems geometric constraints are taken into account like periodic constraint whose involvement is taken to produce a variety of architecturally aesthetic and structurally efficient designs. This paper provides the application of this efficient technology forbridge design and to express its capability in a wider realm of applications.

**Applications to Computer science:**

[Topological data analysis](https://en.wikipedia.org/wiki/Topological_data_analysis) uses techniques from algebraic topology to determine the large scale structure of a set (for instance, determining if a cloud of points is spherical or [toroidal](https://en.wikipedia.org/wiki/Torus)). The main method used by topological data analysis is to:

1. Replace a set of data points with a family of [simplicial complexes](https://en.wikipedia.org/wiki/Simplicial_complex), indexed by a proximity parameter.
2. Analyse these topological complexes via algebraic topology – specifically, via the theory of topology.
3. Encode the persistent homology of a data set in the form of a parameterized version of a [Betti number](https://en.wikipedia.org/wiki/Betti_number), which is called a barcode.

Several branches of [programming language semantics](https://en.wikipedia.org/wiki/Programming_language_semantics), such as [domain theory](https://en.wikipedia.org/wiki/Domain_theory), are formalized

using topology.

In this context, [Steve Vickers](https://en.wikipedia.org/wiki/Steve_Vickers_(computer_scientist)), building on work by [SamsonAbramsky](https://en.wikipedia.org/wiki/Samson_Abramsky) and [Michael B. Smyth](https://en.wikipedia.org/w/index.php?title=Michael_B._Smyth&action=edit&redlink=1), characterizes topological spaces as [Boolean](https://en.wikipedia.org/wiki/Boolean_algebra_(structure)) or [Heyting algebras](https://en.wikipedia.org/wiki/Heyting_algebra) over open sets, which are characterized as [semidecidable](https://en.wikipedia.org/wiki/Semidecidable) (equivalently, finitely observable) properties.

### Appications to Physics:

* Topology is relevant to physics in areas such as [condensed matter physics](https://en.wikipedia.org/wiki/Condensed_matter_physics), [quantum field theory](https://en.wikipedia.org/wiki/Quantum_field_theory) and [physical cosmology](https://en.wikipedia.org/wiki/Physical_cosmology).The topological dependence of mechanical properties in solids is of interest in disciplines of [mechanical engineering](https://en.wikipedia.org/wiki/Mechanical_engineering) and [materials science](https://en.wikipedia.org/wiki/Materials_science). Electrical and mechanical properties depend on the arrangement and network structures of [molecules](https://en.wikipedia.org/wiki/Molecules) and elementary units in materials.[[23]](https://en.wikipedia.org/wiki/Topology#cite_note-23) The [compressive strength](https://en.wikipedia.org/wiki/Compressive_strength) of [crumpled](https://en.wikipedia.org/wiki/Crumpling) topologies is studied in attempts to understand the high strength to weight of such structures that are mostly empty space.

**Applications in multi-body physics:**

* A [topological quantum field theory](https://en.wikipedia.org/wiki/Topological_quantum_field_theory) (or topological field theory or TQFT) is a quantum field theory that computes [topological invariants](https://en.wikipedia.org/wiki/Topological_invariant).
* Although TQFTs were invented by physicists, they are also of mathematical interest, being related to, among other things, [knot theory](https://en.wikipedia.org/wiki/Knot_theory), the theory of [four-manifolds](https://en.wikipedia.org/wiki/Four-manifold) in algebraic topology, and to the theory of [moduli spaces](https://en.wikipedia.org/wiki/Moduli_spaces) in algebraic geometry. [Donaldson](https://en.wikipedia.org/wiki/Simon_Donaldson), [Jones](https://en.wikipedia.org/wiki/Vaughan_Jones), [Witten](https://en.wikipedia.org/wiki/Edward_Witten), and [Kontsevich](https://en.wikipedia.org/wiki/Maxim_Kontsevich) have all won [Fields Medals](https://en.wikipedia.org/wiki/Fields_Medal) for work related to topological field theory.
* The topological classification of [Calabi–Yau manifolds](https://en.wikipedia.org/wiki/Calabi%E2%80%93Yau_manifold) has important implications in [string theory](https://en.wikipedia.org/wiki/String_theory), as different manifolds can sustain different kinds of strings.
* In cosmology, topology can be used to describe the overall shape of the universe.[[26]](https://en.wikipedia.org/wiki/Topology#cite_note-26) This area of research is commonly known as [spacetime topology](https://en.wikipedia.org/wiki/Spacetime_topology).

# Advantages and Disadvantages of Topologies

A network topology refers to the way in which nodes in a network are connected to one another. The network structure defines how they communicate. Each kind of arrangement of the network nodes has its own advantages and disadvantages. Here we tell you about the same.

[**Network topologies**](https://instrumentationtools.com/different-fieldbus-network-topologies/): describe the ways in which the elements of a network are connected. They describe the physical and logical arrangement of network nodes.

Let us look at the advantages different network topologies offer, and their shortfalls.

## ****Bus Topology****

### ****Advantages of Bus Topology:****

* 1.It is easy to set up, handle, and implement.
* 2.It is best-suited for small networks.
* 3.It costs very less.

**Disadvantages of Bus Topology :**

* The cable length is limited. This limits the number of network nodes that can be connected.
* This network topology can perform well only for a limited number of nodes. When the number of devices connected to the bus increases, the efficiency decreases.
* It is suitable for networks with low traffic. High traffic increases load on the bus, and the network efficiency drops.
* It is heavily dependent on the central bus. A fault in the [bus](https://instrumentationtools.com/ethernet-bus-animation/) leads to network failure.
* It is not easy to isolate faults in the network nodes.
* Each device on the network “sees” all the data being transmitted, thus posing a security risk.

## ****Ring Topology****

### ****Advantages of Ring Topology:****

* The data being transmitted between two nodes passes through all the intermediate nodes. A central server is not required for the management of this topology.
* The traffic is unidirectional and the data transmission is high-speed.
* In comparison to a bus, a ring is better at handling load.
* The adding or removing of network nodes is easy, as the process requires changing only two connections.
* The configuration makes it easy to identify faults in network nodes.
* In this topology, each node has the opportunity to transmit data. Thus, it is a very organized network topology.
* It is less costly than a star topology.

**Disadvantages of Ring Topology:**

* The failure of a single node in the network can cause the entire network to fail.
* The movement or changes made to network nodes affect the entire network’s performance.
* Data sent from one node to another has to pass through all the intermediate nodes. This makes the transmission slower in comparison to that in a [star topology](https://instrumentationforum.com/t/star-topology-principle/3749). The transmission speed drops with an increase in the number of nodes.
* There is heavy dependency on the wire connecting the network nodes in the ring.

## ****Mesh Topology****

### ****Advantages of Mesh Topology:****

* The arrangement of the network nodes is such that it is possible to transmit data from one node to many other nodes at the same time.
* The failure of a single node does not cause the entire network to fail as there are alternate paths for data transmission.
* It can handle heavy traffic, as there are dedicated paths between any two network nodes.
* Point-to-point contact between every pair of nodes, makes it easy to identify faults.

**Disadvantages of Mesh Topology:**

* The arrangement wherein every network node is connected to every other node of the network, many connections serve no major purpose. This leads to redundancy of many network connections.
* A lot of cabling is required. Thus, the costs incurred in setup and maintenance are high.
* Owing to its complexity, the administration of a mesh network is difficult.

## ****Star Topology****

### ****Advantages of Star Topology:****

* Due to its centralized nature, the topology offers simplicity of operation.
* It also achieves isolation of each device in the network.
* Adding or removing network nodes is easy, and can be done without affecting the entire network.
* Due to the centralized nature, it is easy to detect faults in the network devices.
* As the analysis of traffic is easy, the topology poses lesser security risk.
* Data packets do not have to pass through many nodes, like in the case of a ring network. Thus, with the use of a high-capacity central hub, traffic load can be handled at fairly decent speeds.

### ****Disadvantages of Star Topology:****

* Network operation depends on the functioning of the central hub. Hence, [central hub](https://instrumentationtools.com/difference-between-router-switch-and-hub/) failure leads to failure of the entire network.
* Also, the number of nodes that can be added, depends on the capacity of the central hub.
* The setup cost is quite high.

## ****Tree Topology****

Imagine a hierarchy of network nodes, with the root node serving client nodes, that in turn serve other lower-level nodes.

The [top-level node](https://instrumentationforum.com/t/tree-topology-principle/3750) is mostly a mainframe computer while other nodes in the hierarchy are mini or microcomputers.

In this arrangement, the node at each level could be forming a star network with the nodes it serves. In this case, the structure combines star and bus topologies and inherits their advantages and disadvantages.

### ****Advantages of Tree Topology:****

* The tree topology is useful in cases where a star or bus cannot be implemented individually. It is most-suited in networking multiple departments of a university or corporation, where each unit (star segment) functions separately, and is also connected with the main node (root node).
* The advantages of centralization that are achieved in a star topology are inherited by the individual star segments in a tree network.
* Each star segment gets a dedicated link from the central bus. Thus, failing of one segment does not affect the rest of the network.
* Fault identification is easy.
* The network can be expanded by the addition of secondary nodes. Thus, scalability is achieved.

### ****Disadvantages of Tree Topology:****

* As multiple segments are connected to a central bus, the network depends heavily on the bus. Its failure affects the entire network.
* Owing to its size and complexity, maintenance is not easy and costs are high. Also, configuration is difficult in comparison to that in other topologies.
* Though it is scalable, the number of nodes that can be added depends on the capacity of the central bus and on the cable type.

## ****Hybrid Topology****

A hybrid topology combines two or more topologies and is meant to reap their advantages.

Obviously, the advantages and disadvantages of a hybrid topology are a combination of the merits and demerits of the topologies used to structure it.

Theorems of topology

# AF+BG theorem

## Statement : Let *F*, *G*, and *H* be [homogeneous polynomials](https://en.wikipedia.org/wiki/Homogeneous_polynomial) in three variables, with *H* having higher degree than *F* and *G*; let *a* = deg *H* − deg *F* and *b* = deg *H* − deg *G* (both positive integers) be the differences of the degrees of the polynomials. Suppose that the [greatest common divisor](https://en.wikipedia.org/wiki/Polynomial_greatest_common_divisor) of *F* and *G* is a constant, which means that the [projective curves](https://en.wikipedia.org/wiki/Projective_curve) that they define in the projective plane P2 have an intersection consisting in a finite number of points. For each point *P* of this intersection, the polynomials *F* and *G* generate an [ideal](https://en.wikipedia.org/wiki/Ideal_(ring_theory)) (*F*, *G*)*P* of the [local ring](https://en.wikipedia.org/wiki/Local_ring) of P2 at *P* (this local ring is the ring of the fractions *n*/*d*, where *n* and *d* are polynomials in three variables and *d*(*P*) ≠ 0). The theorem asserts that, if *H* lies in (*F*, *G*)*P* for every intersection point *P*, then *H* lies in the ideal (*F*, *G*); that is, there are homogeneous polynomials *A* and *B* of degrees *a* and *b*, respectively, such that *H* = *AF* + *BG*. Furthermore, any two choices of *A* differ by a multiple of *G*, and similarly any two choices of *B* differ by a multiple of *F*.

This theorem may be viewed as a generalization of [Bézout's identity](https://en.wikipedia.org/wiki/B%C3%A9zout%27s_identity), which provides a condition under which an integer or a univariate polynomial *h* may be expressed as an element of the [ideal](https://en.wikipedia.org/wiki/Ideal_(ring_theory)) generated by two other integers or univariate polynomials *f* and *g*: such a representation exists exactly when *h* is a multiple of the [greatest common divisor](https://en.wikipedia.org/wiki/Greatest_common_divisor) of *f* and *g*. The AF+BG condition expresses, in terms of [divisors](https://en.wikipedia.org/wiki/Divisor_(algebraic_geometry)) (sets of points, with multiplicities), a similar condition under which a [homogeneous polynomial](https://en.wikipedia.org/wiki/Homogeneous_polynomial) *H* in three variables can be written as an element of the ideal generated by two other polynomials *F* and *G*.

This theorem is also a refinement, for this particular case, of [Hilbert's Nullstellensatz](https://en.wikipedia.org/wiki/Hilbert%27s_Nullstellensatz), which provides a condition expressing that some power of a polynomial *h* (in any number of variables) belongs to the ideal generated by a finite set of polynomials.

# Ehressan’s lemma:

In [mathematics](https://en.wikipedia.org/wiki/Mathematics), or specifically, in [differential topology](https://en.wikipedia.org/wiki/Differential_topology), Ehresmann's lemma or Ehresmann's fibration theorem states that if a [smooth mapping](https://en.wikipedia.org/wiki/Smooth_mapping) {\displaystyle f\colon M\rightarrow N}, where {\displaystyle M} and {\displaystyle N} are [smooth manifolds](https://en.wikipedia.org/wiki/Smooth_manifold), is

1. a [surjective](https://en.wikipedia.org/wiki/Surjective) [submersion](https://en.wikipedia.org/wiki/Submersion_(mathematics)), and
2. a [proper map](https://en.wikipedia.org/wiki/Proper_map), (in particular, this condition is always satisfied if *M* is [compact](https://en.wikipedia.org/wiki/Compact_space)),

then it is a [locally trivial](https://en.wikipedia.org/wiki/Locally_trivial) [fibration](https://en.wikipedia.org/wiki/Fibration). This is a foundational result in [differential topology](https://en.wikipedia.org/wiki/Differential_topology) due to [Charles Ehresmann](https://en.wikipedia.org/wiki/Charles_Ehresmann), and has many variants.

**Whitney embedding theorem:**

In [mathematics](https://en.wikipedia.org/wiki/Mathematics), particularly in [differential topology](https://en.wikipedia.org/wiki/Differential_topology), there are two Whitney embedding theorems, named after [Hassler Whitney](https://en.wikipedia.org/wiki/Hassler_Whitney):

* The strong Whitney embedding theorem states that any [smooth](https://en.wikipedia.org/wiki/Differentiable_manifold) [real](https://en.wikipedia.org/wiki/Real_numbers) *m*-[dimensional](https://en.wikipedia.org/wiki/Dimension_(mathematics)) [manifold](https://en.wikipedia.org/wiki/Manifold) (required also to be [Hausdorff](https://en.wikipedia.org/wiki/Hausdorff_space) and [second-countable](https://en.wikipedia.org/wiki/Second-countable)) can be [smoothly](https://en.wikipedia.org/wiki/Smooth_map) [embedded](https://en.wikipedia.org/wiki/Embedding) in the [real 2*m*-space](https://en.wikipedia.org/wiki/Real_coordinate_space) (R2*m*), if *m* > 0. This is the best linear bound on the smallest-dimensional Euclidean space that all *m*-dimensional manifolds embed in, as the [real projective spaces](https://en.wikipedia.org/wiki/Real_projective_space) of dimension *m* cannot be embedded into real (2*m* − 1)-space if *m* is a [power of two](https://en.wikipedia.org/wiki/Power_of_two) (as can be seen from a [characteristic class](https://en.wikipedia.org/wiki/Characteristic_class) argument, also due to Whitney).
* The weak Whitney embedding theorem states that any continuous function from an *n*-dimensional manifold to an *m*-dimensional manifold may be approximated by a smooth embedding provided *m* > 2*n*. Whitney similarly proved that such a map could be approximated by an [immersion](https://en.wikipedia.org/wiki/Immersion_(mathematics)) provided *m* > 2*n* − 1. This last result is sometimes called the [Whitney immersion theorem](https://en.wikipedia.org/wiki/Whitney_immersion_theorem).

**Proof:** The general outline of the proof is to start with an immersion *f* : *M* → R2*m* with [transverse](https://en.wikipedia.org/wiki/Transversality_(mathematics)) self-intersections. These are known to exist from Whitney's earlier work on the weak immersion theorem. Transversality of the double points follows from a general-position argument. The idea is to then somehow remove all the self-intersections. If *M* has boundary, one can remove the self-intersections simply by isotoping *M* into itself (the isotopy being in the domain of *f*), to a submanifold of *M* that does not contain the double-points. Thus, we are quickly led to the case where *M* has no boundary. Sometimes it is impossible to remove the double-points via an isotopy—consider for example the figure-8 immersion of the circle in the plane. In this case, one needs to introduce a local double point.

Once one has two opposite double points, one constructs a closed loop connecting the two, giving a closed path in R2*m*. Since R2*m* is [simply connected](https://en.wikipedia.org/wiki/Simply_connected), one can assume this path bounds a disc, and provided 2*m* > 4 one can further assume (by the weak Whitney embedding theorem) that the disc is embedded in R2*m* such that it intersects the image of *M* only in its boundary. Whitney then uses the disc to create a [1-parameter family](https://en.wikipedia.org/wiki/Homotopy) of immersions, in effect pushing *M* across the disc, removing the two double points in the process. In the case of the figure-8 immersion with its introduced double-point, the push across move is quite simple.

**Whitney Immersion theorem:**

In [differential topology](https://en.wikipedia.org/wiki/Differential_topology), the **Whitney immersion theorem** (named after [Hassler Whitney](https://en.wikipedia.org/wiki/Hassler_Whitney)) states that for {\displaystyle m>1} any smooth m {\displaystyle m}-dimensional [manifold](https://en.wikipedia.org/wiki/Manifold) (required also to be [Hausdorff](https://en.wikipedia.org/wiki/Hausdorff_space) and [second-countable](https://en.wikipedia.org/wiki/Second-countable)) has a one-to-one [immersion](https://en.wikipedia.org/wiki/Immersion_(mathematics)) in [Euclidean](https://en.wikipedia.org/wiki/Euclidean_space) 2m space ,{\displaystyle 2m}and a (not necessarily one-to-one) immersion in (2m-1)-space .{\displaystyle (2m-1)} Similarly, every smooth m {\displaystyle m}-dimensional manifold can be immersed in the 2m-1 {\displaystyle 2m-1}----dimensional sphere (this removes the m>1 {\displaystyle m>1}constraint).

The weak version, for 2m+1, {\displaystyle 2m+1} is due to [transversality](https://en.wikipedia.org/wiki/Transversality_(mathematics)) ([general position](https://en.wikipedia.org/wiki/General_position), [dimension counting](https://en.wikipedia.org/wiki/Dimension_counting)): two *m*-dimensional manifolds in R ^2m {\displaystyle \mathbf {R} ^{2m}}intersect generically in a 0-dimensional space.

## Statement of the theorem:

The L-genus is the [genus for the multiplicative sequence of polynomials](https://en.wikipedia.org/wiki/Genus_of_a_multiplicative_sequence) associated to the characteristic power series.

x/ tanh(x) =∑ 22k. B2k/(2k)\*x2k =1+x2k–x4/45+……..

{\displaystyle {x \over \tanh(x)}=\sum \_{k\geq 0}{{2^{2k}B\_{2k} \over (2k)!}x^{2k}}=1+{x^{2} \over 3}-{x^{4} \over 45}+\cdots .}

The first two of the resulting L-polynomials are:

* {\displaystyle L\_{1}={\tfrac {1}{3}}p\_{1}}L1= 1/3 p1
* {\displaystyle L\_{2}={\tfrac {1}{45}}(7p\_{2}-p\_{1}^{2})}L2 = 1/45 (7p2– p12)

By taking for the pi{\displaystyle p\_{i}} pp the Pontryagin classes pi (M)  {\displaystyle p\_{i}(M)} of the tangent bundle of a 4*n* dimensional smooth closed oriented manifold M one obtains the L-classes of M. Hirzebruch showed that the n-th L-class of M evaluated on the [fundamental class](https://en.wikipedia.org/wiki/Fundamental_class) of M,( M) {\displaystyle [M]} is equal to σ(M) {\displaystyle \sigma (M)}, the signature of M (i.e. the signature of the intersection form on the 2*n*th cohomology group of M ):

σ(M) = < Ln ( p1 (M)……..,pn (M), [M].>

**Theorem : Limits are not necessarily unique**

Suppose that X has the indiscrete topology and let x ∈ X. Then the constant sequence xn = x converges to y for every y ∈ X.

**Proof:** Suppose U is an open set that contains y. Since X has the indiscrete topology, the only open sets are ∅ and X, so U must be equal to X. This implies that xn ∈ U for all n ≥ 1. In view of the definition of convergence, we thus have xn → y as n → ∞.{\displaystyle \sigma (M)=\langle L\_{n}(p\_{1}(M),\dots ,p\_{n}(M)),[M]\rangle .}

**Theorem :**  **Main facts about closed sets**

1 .If a subset A ⊂ X is closed in X, then every sequence of points of A that converges must converge to a point of A.

2.Both ∅ and X are closed in X.

**Proof:** First, we prove 1 . Suppose {xn} is a convergent sequence of points of A and let x denote its limit. To show that x ∈ A, we assume x ∈ X − A for the sake of contradiction. Then X − A is an open set which contains the limit x, so there is an integer N such that xn ∈ X − A for all n ≥ N. This contradicts our assumption that xn ∈ A for all n ≥ 1. Thus, we must have x ∈ A. Next, we turn to 2 . By definition, the sets ∅, X are both open in X, so their complements X, ∅ are both closed in X. 3

**Theorem : limit point and closure**

Let (X, T) be a topological space and let A ⊂ X. If A0 is the set of all limit points of A, then the closure of A is A = A ∪ A0 .

**Proof:** One has A ⊂ A by definition. To see that A0 ⊂ A as well, suppose that x ∈ A0 . Then every neighbourhood of x intersects A at a point other than x, so x ∈ A. This proves the inclusion A ∪ A 0 ⊂ A. To prove the opposite inclusion, suppose that x ∈ A, but x /∈ A. Then every neighbourhood of x intersects A at a point other than x and so x ∈ A0 .

This proves the opposite inclusion A ⊂ A ∪ A0 .

**Theorem :inclusion maps are continues**

Let (X, T) be a topological space and let A ⊂ X. Then the inclusion map i: A → X defined by i(x) = x is continuous.

**Proof:** Suppose U is an open set in X. Its inverse image is then i −1 (U) = {x ∈ A : i(x) ∈ U} = {x ∈ A : x ∈ U} = A ∩ U. Since U is open in X, the intersection A ∩ U is open in A by the definition of the subspace topology. Thus, it is continuous.

**Theorem : Projection maps are continuous**

Let (X, T) and (Y, T0 ) be topological spaces. If X × Y is equipped with the product topology, then the projection map p1 : X × Y → X defined by p1(x, y) = x is continuous. Moreover, the same is true for the projection map p2 : X × Y → Y defined by p2(x, y) = y.

**Proof:** Given a set U which is open in X, one easily finds that p −1 1 (U) = {(x, y) ∈ X × Y : p1(x, y) ∈ U} = {(x, y) ∈ X × Y : x ∈ U} = U × Y. Since this is open in the product topology of X × Y , the projection map p1 is continuous. Similarly, one has p −1 2 (V ) = X × V for each set V which is open in Y , so p2 is continuous as well.

**Theorem : Some facts about connected spaces.**

1. To say that X is connected is to say that the only subsets of X which are both open and closed in X are the subsets ∅, X.

**Proof**: Suppose A is both open and closed in X, but A is neither empty nor equal to X. Then A and B = X − A are nonempty, open and disjoint with A ∪ B = X, so they form a partition of X. Conversely, suppose A, B form a partition of X. Then A, B are nonempty, open and disjoint with A ∪ B = X. This means that A is neither empty nor equal to X. On the other hand, B = X − A is open in X, so A is both open and closed in X.

**Theorem** : **Some facts about connected spaces** .

1. The continuous image of a connected space is connected: if X is connected and f : X → Y is continuous, then f(X) is connected.

**Proof:** We may assume that f(X) = Y without loss of generality. Suppose that A, B form a partition of Y . Then A, B are nonempty, open and disjoint with A ∪ B = Y . Since f is continuous, it easily follows that the inverse images U = f −1 (A), V = f −1 (B) are nonempty, open and disjoint with U ∪ V = X. In other words, they form a partition of X, which is contrary to the fact that X is connected. This implies that Y must be connected as well.

**Theorem** : **Some facts about connected spaces** .

**3 .** A subset of R is connected if and only if it is an interval.

**Proof, part 1:** An interval is a set I ⊂ R which contains all points between inf I and sup I. Suppose that A ⊂ R is a set which is not an interval. Then there exists some real number x such that inf A < x < sup A, x /∈ A. Since x is larger than the greatest lower bound of A, we see that x is not a lower bound of A, so a < x for some a ∈ A. Similarly, we must also have x < b for some b ∈ A. It easily follows that the sets U = (−∞, x) ∩ A, V = (x, +∞) ∩ A are nonempty, disjoint and open in A with U ∪ V = A. Thus, these sets form a partition of A and so A is not connected.

**Theorem** : **Some facts about connected spaces**.

**3** . A subset of R is connected if and only if it is an interval.

**Proof, part 2:** Conversely, suppose that I ⊂ R is an interval which has a partition U|V . Pick two points x ∈ U and y ∈ V . Assuming that x < y without loss of generality, we now set U 0 = [x, y] ∩ U, z = sup U 0 . Given any integer n ∈ N, there exists a point xn ∈ U 0 such that z − 1/n ≤ xn ≤ z. Since U 0 is closed in U and xn → z as n → ∞, we see that z ∈ U 0 . In particular, z ∈ U and x ≤ z < y. Since U is open in I, we must have z + ε ∈ U for all small enough ε > 0 and so z + ε ∈ U 0 for all small enough ε > 0. This contradicts the fact that z = supU 0 .

**Theorem** : **Some facts about connected** **spaces.**

1. If a connected space A is a subset of X and the sets U, V form a partition of X, then A must lie entirely within either U or V .

**Proof**: Consider the sets A ∩ U and A ∩ V . These are open in A, they are disjoint and their union is equal to (A ∩ U) ∪ (A ∩ V ) = A ∩ (U ∪ V ) = A ∩ X = A. Since A is connected, one of the two sets must be empty. Suppose that A ∩ U is empty, as the other case is similar. Then A is a subset of X = U ∪ V which does not intersect U, so A ⊂ V .

**Theorem** : **Some more facts about connected spaces** .

If A is a connected subset of X, then A is connected as well.

**Proof:** Suppose U, V form a partition of the closure A. Since A is a connected subset of this partition, it must lie within either U or V . Assume that A ⊂ U without loss of generality. Then we have A ⊂ U ⊂ X − V and this makes X − V a closed set that contains A. In view of the definition of the closure, the smallest such set is A, so A ⊂ X − V =⇒ U ∪ V ⊂ X − V. This means that V must be empty, which is contrary to assumption. In particular, A has no partition and the result follows.

**Theorem: Some more facts about connected spaces**.

**2** . Consider a collection of connected sets Ui that have a point in common. Then the union of these sets is connected as well.

**Proof:** Suppose A, B form a partition of the union and let x be a point which is contained in Ui for all i. Then x belongs to either A or B. Assume that x ∈ A without loss of generality. Since Ui is a connected subset of the partition, it must lie entirely within either A or B. Since Ui contains x, however, we must have Ui ⊂ A for all i =⇒ [ i Ui ⊂ A. This means that B must be empty, which is contrary to assumption. In particular, S i Ui has no partition and the result follows.

**Theorem : Main facts about compact spaces**.

A closed subset of a compact space is compact.

**Proof:**  Suppose X is compact and let A ⊂ X be closed. To show that A is compact, suppose the sets Ui form an open cover of A. Adjoining X − A to these sets gives an open cover of X. This has a finite subcover by compactness, so X is covered by finitely many of the sets Ui along with X − A. In particular, A itself is covered by finitely many of the sets Ui and so A is compact